SPHERE THEOREM FOR MANIFOLDS WITH POSITIVE CURVATURE

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Abstract. In this paper, we prove that, for any integer $n \geq 2$, there exists an $\epsilon_n \geq 0$ so that if M is an n-dimensional complete manifold with sectional curvature $K_M \geq 1$ and if M has conjugate radius bigger than $\frac{\pi}{2}$ and contains a geodesic loop of length $2(\pi - \epsilon_n)$, then M is diffeomorphic to the Euclidian unit sphere S^n.

1. Introduction

One of the fundamental problems in Riemannian geometry is to determine the relation between the topology and the geometry of a Riemannian manifold. In this way the Toponogov's theorem and the critical point theory play an important rule. Let M be a complete Riemannian manifold and fix a point p in M and define $d_p(x) = d(p, x)$. A point $q \neq p$ is called a critical point of d_p or simply of the point p if, for any nonzero vector $v \in T_qM$,there exists a minimal geodesic γ joining q to p such that the angle $(v, \gamma'(0) \leq \frac{\pi}{2}$. Suppose M is an n-dimensional complete Riemannian manifold with sectional curvature $K_M \geq 1$. By Myers' theorem the diameter of M is bounded from above by π. In [Ch] Cheng showed that the maximal value π is attained if and only if M is isometric to the standard sphere. It was proved by Grove and Shiohama [GS] that if $K_M \geq 1$ and the diameter of M $diam(M) > \frac{\pi}{2}$ then M is homeomorphic to a sphere.

Hence the problem of removing homeomorphism to diffeomorphism or finding conditions to guarantee the diffeomorphism is of particular interest. In [Xi3] C. Xia showed that if $K_M \geq 1$ and the conjugate radius of M $\rho(M) > \pi/2$ and if M contains a geodesic loop of length 2π then M is isometric to $S^n(1)$.

1.1. **Definition.** *Let M be an n-dimensional Riemannian manifold and p be a point in M. Let Conj(p) denote the set of first conjugate points to p on all geodesics issuing from p. The conjugate radius* $\rho(p)$ *of* M

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at p is defined as

$$\rho(p) = d(p, Conj(p)) \qquad \text{if } Conj(p) \neq \emptyset$$

and

$$\rho(p) = +\infty \qquad \text{if } Conj(p) = \emptyset$$

Then the conjugate radius of M is:

$$\rho(M) = \inf_{x \in M} \rho(x).$$

Many interesting results have been proved by using the critical points theory and Toponogov's theorem [C], [GS], [Pe], [S], [Sh], [SS], [Xi1], [Xi2], [Xi3]. In otherewise J. Cheeger and T. Colding in [CC] have proven the following

Theorem A. *There exists a number, $\epsilon(n) > 0$, depending only on the integer n such that, for any two Riemannian manifolds Z_1, Z_2, if $d_{GH}(Z_1, Z_2) < \epsilon(n)$, then Z_1 and Z_2 are diffeomorphic.* where $d_{GH}(Z_1, Z_2$ denote the Gromov-Hausdroff distance.

The purpose of this paper is to prove the following :

1.2. **Theorem.** *For any $n \geq 2$ there exists a positif constant $\epsilon(n)$ depending only on n such that for any $\epsilon \leq \epsilon(n)$, if M is an n-dimensional complete connected Riemannian manifold with sectional curvature $K_M \geq 1$ and conjugate radius $\rho(M) > \frac{\pi}{2}$ and if M contains a geodesic loop of length $2(\pi - \epsilon)$ then M is diffeomorphic to an n-dimensional unit sphere $S^n(1)$.*

2. Proof

Since $K_M \geq 1$, M is compact. Let $i(M)$ denote the injectivity radius of M.

By definition we have

$$i(M) = \inf_{x \in M} d(x, C(x)).$$

Since M is compact and the function $x \mapsto d(x, C(x))$ is continuous, there exists $p \in M$ such that $i(M) = d(p, C(p))$. Since $C(p)$ is compact there exists $q \in C(p)$ such that $i(M) = d(p, q) = d(p, C(p))$. Then

a) either there exists a minimal geodesic σ joining p to q such that q is a conjugate point of p or

b) there exists two minimal geodesics σ_1 and σ_2 joining p to q such that $\sigma_1'(l) = -\sigma_2'(l), \quad l = d(p, q)$. See [C].

2.1. **Lemma.** *Let M be an n-dimensional complete, connected Riemannian manifold with sectional curvature $K_M \geq 1$ and the conjugate radius $\rho(M) > \frac{\pi}{2}$, then $i(M) > \frac{\pi}{2}$.*

Proof of the lemma

If $a)$ holds, then $i(M) = d(p,q) > \pi/2$.

Suppose $b)$ holds . Since $q \in C(p)$, we have $p \in C(q)$ and consequently $d(p,q) = d(q, C(q))$. This implies that $\sigma_1'(0) = -\sigma_2'(0)$. Set $D(x) = \max_{y \in M} d(x,y)$. Then

$$D(x) \geq \max_{y \in C(x)} d(x,y) \geq \rho(M) > \pi/2.$$

Since M is compact, there exist a point $y \in M$ such that $D(x) = d(x,y) > \pi/2$ and y is the unique farthest point and the critical one for the distance $d(x,.)$. Set $A(x) = y$; thus we define a continuous map $A : M \mapsto M$ (see [Xi3]. By the Berger-Klingenberg theorem, M is homeomorphic to the unit sphere $S^n(1)$ and since $A(x) \neq x$ for all $x \in M$ the Brouwer fixed point theorem sets that the degree of A is $(-1)^{n+1}$ and consequently A is surjective. Let $r \in M$ the point so that $p = A(r)$. Hence $d(p,r) > \pi/2$. If $r = q$ then $i(M) = d(p,q) > \pi/2$ otherwise there exists a minimal geodesic from q to r, σ_3 such that

$$\measuredangle(\sigma_3'(0), -\sigma_1'(l)) \leq \pi/2 \quad or \quad \measuredangle(\sigma_3'(0), -\sigma_2'(l)) \leq \pi/2.$$

Suppose $\measuredangle(\sigma_3'(0), -\sigma_1'(l)) \leq \pi/2$. Applying the Toponogov's theorem [T] to the hinge (σ_1, σ_3), we have:

$$\cos d(p,r) \geq \cos d(p,q) \cos d(q,r) + \sin d(p,q) \sin d(q,r) \cos \angle(\sigma_3'(0), -\sigma_1'(l))$$

$$\geq cos d(p,q) \cos d(q,r). \tag{1}$$

Since r is far from p in the sense that $d(p,r) > \pi/2$ then r is near to q i.e $d(q,r) < \pi/2$ and from (1) we have

$$\cos d(p,q) < 0$$

and consequently

$$i(M) = d(p,q) > \pi/2$$

which proves the lemma.

2.2. **Lemma.** *Let M be a complete connected n-dimensional Riemannian manifold with sectional curvature $K_M \geq 1$ and conjugate radius $\rho(M) > \frac{\pi}{2}$. If M contains a geodesic loop of length at least $2(\pi - \epsilon)$ then $diam(M) \geq \pi - \tau(\epsilon)$ where $\tau(\epsilon) \mapsto 0$ when $\epsilon \mapsto 0$.*

Proof

Since $i(M) > \pi/2$ then there exist $\delta > 0$ such that $i(M) > \pi/2 + \delta$. Let γ be a loop with length $2\pi - 2\epsilon$. Let $x = \gamma(0) = \gamma(2\pi - 2\epsilon)$, $y = \gamma(\pi/2 + \delta)$, $m = \gamma(\pi - \epsilon)$ and $z = \gamma(\frac{3(\pi-\epsilon)}{2} - \delta)$

Let

$$\gamma_1 = \gamma_{/[0,\frac{\pi}{2}+\delta]}, \quad \gamma_2 = \gamma_{/[\frac{\pi}{2}+\delta,\pi-\epsilon]} \quad \gamma_3 = \gamma_{/[\pi-\epsilon,\frac{3(\pi-\epsilon)}{2}-\delta]}$$

and $\gamma_4 = \gamma_{/[\frac{3(\pi-\epsilon)}{2}-\delta,2\pi-2\epsilon]}$.

Then the geodesics γ_i are minimal. Let σ be a minimal geodesic joining m and x.

We claim that $L(\sigma) \geq \pi - \tau(\epsilon)$.

Set $\alpha = \measuredangle(\sigma'(0), -\gamma'(\pi-\epsilon))$ and $\beta = \measuredangle(\sigma'(0), \gamma'(\pi-\epsilon))$. Applying the Toponogov's theorem to the triangles $(\gamma_1, \gamma_2, \sigma)$ and $(\gamma_3, \gamma_4, \sigma)$ respectively, one can take two triangles $(\overline{\gamma}_1, \overline{\gamma}_2, \overline{\sigma})$ and $(\overline{\gamma}_3, \overline{\gamma}_4, \overline{\sigma})$ on the unit sphere $S^2(1)$ with vertices $\overline{x}, \overline{y}, \overline{m}$ and $\overline{x}, \overline{z}, \overline{m}$ respectively satisfying:

$$L(\overline{\gamma}_i) = L(\gamma_i), \quad i = 1,2,3,4; L(\overline{\sigma}) = L(\sigma);$$

hence $\overline{\alpha} \leq \alpha, \overline{\beta} \leq \beta$ where $\overline{\alpha}$ and $\overline{\beta}$ are the angles at $\overline{m}$ of the triangles $(\overline{\gamma}_1, \overline{\gamma}_2, \overline{\sigma})$ and $(\overline{\gamma}_3, \overline{\gamma}_4, \overline{\sigma})$ respectively. We have:$\alpha \leq \pi/2$ or $\beta \leq \pi/2$. Suppose, without lost the generality, that $\alpha \leq \pi/2$. Let $\overline{x}'$ be the antipodal point of $\overline{x}$ on the sphere $S^2(1)$ and $\overline{\sigma}_1$ the minimal geodesic from $\overline{m}$ to $\overline{x}'$. If $\overline{\alpha}'$ and $\overline{\beta}'$ are the angles at $\overline{m}$, of triangles $(\overline{y}, \overline{m}, \overline{x}')$ and $(\overline{z}, \overline{m}, \overline{x}')$ respectively then we have:

$$\begin{aligned} d((\overline{y}, \overline{x}') &= \pi - d(\overline{y}, \overline{x}) \\ &= \pi - d(x,y) = \frac{\pi}{2} - \delta \end{aligned} \tag{2}$$

Hence, using the trigonometric law on the triangle $(\overline{y}, \overline{m}, \overline{x}')$ we have:

$$\begin{aligned} \sin d(\overline{m}, \overline{y}) \sin d(\overline{m}, \overline{x}') \cos \overline{\alpha}' &= \cos d(\overline{y}, \overline{x}') - \cos d(\overline{m}, \overline{y}). \cos d(\overline{m}, \overline{x}') \\ &= \cos(\frac{\pi}{2} - \delta) - \cos(\frac{\pi}{2} - \delta - \epsilon). \cos d(\overline{m}, \overline{x}') \\ &= \sin(\delta) - \sin(\delta + \epsilon) \cos d(\overline{m}, \overline{x}') \leq 0 \end{aligned}$$

which means that

$$\cos d(\overline{m}, \overline{x}') \geq \frac{\sin(\delta)}{\sin(\delta+\epsilon)}.$$

It follows that

$$L(\sigma) = d(\overline{m}, \overline{x}') \leq \cos^{-1}\left(\frac{\sin(\delta)}{\sin(\delta+\epsilon)}\right) = \tau(\epsilon)$$

with $\tau(\epsilon) \mapsto 0$ when $\epsilon \mapsto 0$.

Hence

$$d(m,x) = d(\overline{m}, \overline{x}) \geq \pi - \tau(\epsilon).$$

2.3. **Lemma.** *Let M be a complete connected n-dimensional Riemannian manifold with sectional curvature $K_M \geq 1$ and $diam(M) \geq \pi - \epsilon$, then for all $x \in M$ there exists a point x' such that $d(x, x') \geq \pi - \Gamma(\epsilon)$ hence $Rad(M) \geq \pi - \Gamma(\epsilon)$ with $\Gamma(\epsilon) \mapsto 0$ when $\epsilon \mapsto 0$ where*

$$Rad(M) = min_{x \in M} max_{y \in M} d(x, y)$$

Proof

Since M is compact, its injectivity radius $i(M)$ is positive and set r_0 a positive number not larger than $i(M)$. Let p, q be two points in M such that $d(p, q) = diam(M) \geq \pi - \epsilon$. Let $x \in M$ and suppose $x \neq p$ and $x \neq q$. Consider the triangle (p, q, x) and let y be a point of the segment $[q, x]$ such that $d(q, y) = \frac{r}{2}$ with $r < r_0$ and $d(p, x) > 2r$ $d(q, x) > 2r$. For any point $s \in B(y, r)$ the function

$$z \mapsto e_{yz}(s) = d(y, s) + d(z, s) - d(y, z)$$

is continuous and if z is on the prolongation of the geodesic joining y to s we have: $e_{yz}(s) = 0$ (this is possible since $r < r_0$).

Hence the function $z \mapsto e_{yz}(q)$ is continuous on the sphere $S(y, r)$, and consequently there exists $z \in S(y, r)$ such that

$$e_{yz}(q) = d(y, q) + d(z, q) - d(y, z) < \epsilon \tag{3}$$

For any point $v \in M$ we have:

$$d(p, v) + d(q, v) + d(p, q) \leq 2\pi$$

hence

$$|d(p, v) + d(q, v) - \pi| \leq \epsilon. \tag{4}$$

Let $\overline{p}'$ be the antipodal point of $\overline{p}$ on the sphere S^2.

$$d(\overline{p}, \overline{q}) = d(p, q);$$

$$d(\overline{z}, \overline{p}') = \pi - d(\overline{z}, \overline{p}) = \pi - d(p, z).$$

We have

$$|d(\overline{q}, \overline{y} - d(\overline{y}, \overline{p}')| = |d(\overline{q}, \overline{y}) - \pi + d(\overline{p}, \overline{y})| < \epsilon.$$

In the same way, we have:

$$|d(\overline{q}, \overline{z}) - \pi + d(\overline{p}, \overline{z})| = |d(q, z) - \pi + d(p, z)| < \epsilon$$

Hence

$$\begin{aligned} d(\overline{y}, \overline{z}) = d(y, z) d(q, y) + d(q, z) - \epsilon \geq \pi - \epsilon - d(p, y) + d(q, z) - \epsilon \\ \geq \pi - d(\overline{p}, \overline{y}) + d(\overline{q}, \overline{z}) - 2\epsilon \geq d(\overline{p}', \overline{y}) + d(\overline{q}, \overline{z}) - 2\epsilon \end{aligned} \tag{5}$$

$$\geq d(\overline{p}', \overline{y}) + d(\overline{p}', \overline{z}) - 3\epsilon \tag{6}$$

Thus $d(\overline{p}', \overline{y}) > \frac{r}{2} - \frac{3}{2}\epsilon$ and $d(\overline{p}', \overline{z}) > \frac{r}{2} - \frac{3}{2}\epsilon$.

Suppose $d(\overline{p}', \overline{y}) \leq \frac{r}{2} - \frac{3}{2}\epsilon$ then $d(\overline{p}, \overline{y}) \geq \pi - \frac{r}{2} + \frac{3}{2}\epsilon$ which contradicts (4).

Let

$$\overline{l}_1' = d(\overline{p}', \overline{y}); \overline{l}_2' = d(\overline{p}', \overline{z}) \text{ and } \overline{l}_0' = d(\overline{z}, \overline{y}).$$

Applying the Topogonov's theorem to the triangle $(\overline{y}, \overline{p}', \overline{z})$, we have:

$$\sin \overline{l}_1' \sin \overline{l}_2' \cos \measuredangle \overline{p}' = \cos \overline{l}_0' - \cos \overline{l}_1' \cos \overline{l}_2'$$

From inequalty (6) we get:

$$\sin \overline{l}_1' \sin \overline{l}_2' \cos \measuredangle \overline{p}' < \cos(\overline{l}_1' + \overline{l}_2' - 3\epsilon) - \cos \overline{l}_1' \cos \overline{l}'2$$

$$< -\cos \overline{l}_1' \cos \overline{l}_2' (1 - \cos 3\epsilon) - \sin \overline{l}_1' \sin \overline{l}_2' \cos 3\epsilon - \sin(\overline{l}_1' + \overline{l}_2') \sin 3\epsilon$$

Hence

$$cos \measuredangle \overline{p}' < -\cos 3\epsilon + (cotg \overline{l}_2' + cotg \overline{l}_1') \sin 3\epsilon - cotg \overline{l}_1' . cotg \overline{l}_2' (1 - \cos 3\epsilon).$$

Thus $\measuredangle \overline{p}' > \pi - \tau_1(\epsilon)$ with

$$\tau_1(\epsilon) = \cos^{-1} \Big(cos 3\epsilon - (cotg \overline{l}_2' + cotg \overline{l}_1') \sin 3\epsilon + cotg \overline{l}_1' . cotg \overline{l}_2' (1 - \cos 3\epsilon) \Big)$$

and $\tau_1(\epsilon) \mapsto 0$ as $\epsilon \mapsto 0$.

The trigonometric law on the sphere shows that the angle at $\overline{p}'$ of the triangle $(\overline{y}, \overline{p}', \overline{z})$ is equal to the angle at $\overline{p}$ of the triangle $(\overline{y}, \overline{p}, \overline{z})$ which is not bigger than the angle at p of the triangle (y, p, z) in M.

Hence $\measuredangle(y, p, z) > \pi - \tau_1(\epsilon)$.

By applying the relation (4) to x and y we have:

$$-\epsilon \leq d(p, x) + d(q, x) - \pi \leq \epsilon, \tag{7}$$

$$-\epsilon \leq d(p, y) + d(q, y) - \pi \leq \epsilon. \tag{8}$$

Since $y \in [q, r]$, we conclude from (7) and (8) that $e_{py}(x) < 2\epsilon$, which shows that the angle at x of the triangle (p, x, y) is close to π and consequently its angle at p is small. there exists

$\tau_2(\epsilon)$ such that $\measuredangle p \leq \tau_2(\epsilon)$ where $\tau_2(\epsilon) \mapsto 0$ as $\epsilon \mapsto 0$.

Take r small enough; then $d(p, x) < d(p, y)$; otherwise $d(p, x)$ is close to π and we conclude by taking $x' = p$.

Let $\tilde{x} \in [p, y]$ such that $d(p, \tilde{x}) = d(p, x)$; then

$$d(x, \tilde{x}) \leq \pi \measuredangle(y, p, x) = \tau_3(\epsilon). \tag{9}$$

Since
$$d(p,z)+d(z,q)\geq d(p,q)\geq \pi-\epsilon$$
and
$$d(y,z)=r>d(q,y)+d(q,z)-\epsilon$$
we have:

if $d(p,x)\leq d(q,z)+\epsilon$ then $r<\frac{2}{3}\epsilon$ and it suffices to take $x'=q$ and $\Gamma(\epsilon)\leq\frac{7}{3}\epsilon$; if $d(p,x)>d(q,z)+\epsilon$ then
$$d(p,z)\geq \pi-d(q,z)-\epsilon>\pi-d(p,x)$$
hence there exists a point $x'\in[p,z]$ such that
$$d(p,x')=\pi-d(p,x).$$

It suffices to show that $d(x',\tilde{x})\geq\pi-\tau_4(\epsilon)$.

Applying the Toponogov theorem to the triangle $(\tilde{x},p,x')$, we get

$$\cos d(x',\tilde{x})=\cos d(p,x')\cos d(p,\tilde{x})+\sin d(p,x')\sin d(p,\tilde{x})\cos\measuredangle p$$
$$=-\cos^2 d(p,\tilde{x})+\sin^2 d(p,\tilde{x})\cos\measuredangle p$$

Since $x'\in[p,z]$ and $\tilde{x}\in[p,y]$ the angle at $\overline{p}$ of the triangle $(\overline{y},\overline{p},\overline{z})$ is less or equal to the angle at $\overline{p}$ of the triangle $(\overline{\tilde{x}},\overline{p},\overline{x'})$ which is not bigger than the angle at p of triangle $(\tilde{x},p,x')$ in M.

$$\measuredangle(\tilde{x},p,x')\geq\measuredangle(\overline{\tilde{x}},\overline{p},\overline{x'}\geq\measuredangle(\overline{y},\overline{p},\overline{z})\geq\pi-\tau_4(\epsilon).$$

Hence
$$cosd(x',\tilde{x})\leq-\cos^2 d(p,\tilde{x})+\sin^2 d(p,\tilde{x})\cos(\pi-\tau_4(\epsilon)$$
$$=-\cos(\tau_4(\epsilon))-\cos^2 d(p,x)\,(1-\cos(\tau_4(\epsilon)))\leq-\cos(\tau_4(\epsilon))$$
$$\Rightarrow d(\tilde{x},x')>\pi-\tau_4(\epsilon).$$

From the triangle inequality and the inequality (9) we have
$$d(x,x')\geq d(\tilde{x},x')-d(\tilde{x},x)\geq\pi-\tau_5$$
Thus, lemma 2.3 follows.

In [Gr] M. Gromov generalized the classic notion of Hausdorff distance between two compact subsets of the same metric space. He considered the set of compact Riemannian manifolds as a subset of the set of all compact metric spaces.

2.4. **Definitions.** *1) Let X, Y be two metric espaces; a map $f : X \longrightarrow Y$ is said to be an ϵ-approximation if the image set $f(X)$ is ϵ-dense in Y and, for any $x, y \in X$, $|d(f(x), f(y)) - d(x, y)| < \epsilon$.*

2) The Gromov-Hausdorff distance $d_{GH}(X, Y)$ between X and Y is the infimum of values of $\epsilon > 0$ such that there exist ϵ- approximations $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$. In [Co1] and [Co2] Colding showed the two equivalent conditions:

1) $Rad(M) \geq \pi - \epsilon$

2) $d_{GH}(M, S^n(1)) \leq \tau_5(\epsilon)$ with $\tau_5(\epsilon) \mapsto 0$ as $\epsilon \mapsto 0$.

Then theorem 1.2 follows from these conditions and the theorem A.

3. References

[C] J.Cheeger, Critical points of distance functions and applications to geometry , Lectures notes 1504 (1991) 1-38.

[CC] J.Cheeger and T. Colding, On the structure of spaces with Ricci curvature bounded from below, J. Diff. Geom. 46 (1997) 406-480.

[Ch] S.Y.Cheng, Eigenvalue Comparison Theorem and Geometric Applications Math.Z.143(1975) p. 289-297.

[Co1] T.H. Colding, Shape of manifolds with positive Ricci curvature, Invent. Math. 124 (1996), 175-191

[Co2] T.H. Colding, Large manifolds with positive Ricci curvature, Invent. Math. 124 (1996), 193-214

[Gr] M.Gromov, Structures métriques pour les variétés Riemanniennes, Rédigé par J.Lafontaine et P.Pansu,cedic 1981.

[GS] K. Grove and K. Shiohama, A generalized sphere theorem, Ann. Math. 106 (1977), 201-211.

[Pe] P. Pertersen, Comparison geometry problem list, Riemannian geometry (Waterloo, ON,(1993) 87-115, fields Inst. Monogr., 4, Amer. Math. Soc., Providence, RI 1996.

[S] K. Shiohama, A sphere theorem for manifolds of positive Ricci curvature, Trans. Amer. Math. Soc. 275 n.2 (1983) 811-819.

[Sh] Z. Shen , Complete manifolds with nonnegative Ricci curvature and large volume growth, Invent. Math. 125 (1996) 393-404.

[SS] J. Sha and Z. Shen, Complete manifolds with nonnegative curvature and quadratically nonnegative curved infinity, Amer. J. Math. 119 (1997) 1399-1404.

[T] V.A.Toponogov, Computation of length of a closed geodesic on convex surface, Dokl. akad. Nank SSSR 124 (1959) 282-284.

[Xi1] C.Y. Xia , Large volume growth and topology of open manifolds, Math. Z. 239 (2002) 515-526.

[Xi2] " " , Complete manifolds with sectional curvature bounded below

and large volume growth, Bull. London Math. Soc. 34 (2002) 229-235.
[Xi3] " " , Some applications of critical point theory of distance functions on Riemannian manifolds, Compos.math.132, n.1(2002)pp 49-55.

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